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Some spectral equivalences between Schrödinger operators

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Abstract

Spectral equivalences of the quasi-exactly solvable sectors of two classes of Schrödinger operators are established, using Gaudin-type Bethe ansatz equations. In some instances the results can be extended leading to full isospectrality. In this manner we obtain equivalences between PT -symmetric problems and Hermitian problems. We also find equivalences between some classes of Hermitian operators.

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1. Introduction

In recent years there has been substantial activity studying the relationships between integrable systems, which can be exactly solved via Bethe ansatz methods, and the spectra of differential equations [1–4]. See the review article [5] for further references. This has led to a deep understanding of the spectral properties of certain Schrödinger operators which are not Hermitian, but possess the more general property of PT symmetry [6, 7]. An initially surprising result was the establishment of reality for the energy levels of particular non-Hermitian operators with PT symmetry [8, 9]. An alternative approach to prove reality of a PT -symmetric operator is to construct an equivalent Hermitian Hamiltonian [10–13]. Such a construction has proven difficult and only a few explicit (non-perturbative) results are known [14–20]. For a detailed review of the field we refer to [5, 21, 22].

Our goal here is to further expose spectral equivalences between Schrödinger operators which are quasi-exactly solvable (QES) [23, 24], i.e. operators for which part of the spectrum can be determined algebraically. We will consider two cases, namely one associated with sextic potentials with an angular momentum like term, and another where the potentials are expressed in terms of hyperbolic functions. For both cases the starting point is to begin with a Hamiltonian which admits two exact Bethe ansatz solutions. Here, the Bethe ansatz solutions

are of the Gaudin (additive) form with finitely many roots, so our approach is different from the cases [1–4, 8, 9] for (multiplicative) Bethe ansatz equations with an infinite number of roots. Each solution can be mapped to the QES sector of a one-dimensional Schrödinger operator, and these turn out to have different potentials. Since the QES eigenvalues of the Schrödinger operators are the same as the Hamiltonian, this establishes a spectral equivalence at the level of the QES sectors.

For the sextic case our results show there is a spectral equivalence of the QES levels between certain Hermitian and a PT -symmetric potentials. This provides a starting point to determine equivalences in a more general context, which we also discuss. In some instances we can establish these equivalences rigorously, using the techniques of Bender–Dunne polynomials [25] and Darboux–Crum [26, 27] transformations. In other cases we give conjectures which we find are supported by numerical calculation of the spectra.

In the last section of the paper we study a second class of Hamiltonians from which we can determine spectral equivalences of the QES levels for hyperbolic potentials. Here we find that the equivalence exists between two Hermitian potentials or alternatively, by use of a unitary transformation, two PT -symmetric potentials. This contrasts the case of the sextic potentials where the equivalence is between Hermitian and PT -symmetric potentials. For the normalizable cases, we obtain a complete spectral equivalence by showing that the two potentials are supersymmetric partners [28, 29].

2. Spectral equivalences in sextic potentials

We begin by considering a class of Hamiltonians given by

$$H_p = A_p S_p^z + B_p (d_p^\dagger S_p^- + \xi_p d_p S_p^+), \tag{1}$$

where $\{d_p, d_p^\dagger\}$ are boson operators and $\{S_p^z, S_p^+, S_p^-\}$ are $su(2)$ operators, while $\xi_p = \pm 1$ is a discrete variable. Through use of the algebraic Bethe ansatz (ABA) method in the *quasi-classical limit* (see, for example, [30]), the energies and corresponding Bethe ansatz equations (BAE) are found to be

$$E_p = A_p (M_p + \kappa_p) + \xi_p B_p \sum_{j=1}^{M_p} v_p^{(j)}, \tag{2}$$

$$\frac{2\kappa_p}{v_p^{(j)}} + \xi_p v_p^{(j)} + \frac{A_p}{B_p} = \sum_{k \neq j}^{M_p} \frac{2}{v_p^{(k)} - v_p^{(j)}}. \tag{3}$$

The parameter κ_p is determined by the reference state $|\phi_p\rangle$ which satisfies

$$S_p^z |\phi_p\rangle = \kappa_p |\phi_p\rangle, \quad S_p^- |\phi_p\rangle = 0, \quad d_p |\phi_p\rangle = 0,$$

and the eigenstates have the form

$$|\{v_p^{(k)}\}\rangle = \prod_{j=1}^{M_p} (v_p^{(j)} d_p^\dagger + S_p^+) |\phi_p\rangle. \tag{4}$$

2.1. Equivalences of QES sectors

Next we consider the Hamiltonian

$$H = \epsilon(n_a - n_b - n_c) + \Omega(a^\dagger bc + ab^\dagger c^\dagger) \tag{5}$$

where $\{\alpha, \alpha^\dagger : \alpha = a, b, c\}$ are canonical boson operators with the usual number operators $n_\alpha = \alpha^\dagger \alpha$. It is straightforward to verify that this Hamiltonian commutes with the conserved operators $N = 2n_a + n_b + n_c$, $K = n_b - n_c$. To make a connection with (1), we make the following two realizations of $su(2)$ operators,

$$\begin{aligned} S_1^+ &= a^\dagger c, & S_1^- &= ac^\dagger, & S_1^z &= \frac{1}{2}(n_a - n_c), \\ S_2^+ &= -b^\dagger c^\dagger, & S_2^- &= bc, & S_2^z &= \frac{1}{2}(n_b + n_c + 1) \end{aligned} \quad (6)$$

and set

$$d_1^\dagger = b^\dagger, \quad d_2^\dagger = a^\dagger.$$

We may now express the Hamiltonian (5) as

$$H = H_1 - \frac{\epsilon}{4}(N + 3K) = H_2 + \frac{\epsilon}{2}(N + 3I), \quad (7)$$

where

$$A_1 = -A_2 = 3\epsilon, \quad B_1 = B_2 = \Omega, \quad \xi_1 = -\xi_2 = 1. \quad (8)$$

Hereafter we set $\Omega = 1$.

For the reference states we choose $|\phi_p\rangle = |\phi(q_p)\rangle$

$$|\phi(q_p)\rangle = \frac{1}{\sqrt{q_p!}}(c^\dagger)^{q_p}|0\rangle,$$

which leads to the values

$$\kappa_1 = -q_1/2, \quad \kappa_2 = (q_2 + 1)/2. \quad (9)$$

Given a solution for H with the M_1 Bethe roots $\{v_1^{(j)}\}$ associated with H_1 and q_1 , we need to determine the relationship to the solution of the M_2 Bethe roots $\{v_2^{(j)}\}$ for H_2 with q_2 . From the form of the eigenstates (4) we can deduce the values of the conserved operators

$$N = M_1 + q_1 = 2M_2 + q_2, \quad K = M_1 - q_1 = -q_2.$$

Solving this gives

$$q_1 = M_2 + q_2, \quad M_1 = M_2. \quad (10)$$

This imposes the restriction $M_1 \leq q_1$. (Note that for $M_1 > q_1$ the eigenstate (4) vanishes.) For convenience set $q_2 = q$ and $M_2 = M$. If E is the energy corresponding to the Hamiltonian (5) then through (7) we have

$$\begin{aligned} E &= E_1 - \frac{\epsilon}{2}(M - q) \\ &= E_2 + \frac{\epsilon}{2}(2M + q + 3). \end{aligned} \quad (11)$$

At this point we remark that the energy expression (2) and associated Bethe ansatz equations (3) have precisely the form of the exact solution for the QES Schrödinger operator with sextic potential [24]. Explicitly,

$$\begin{aligned} -\psi_p''(x) + V_p(x)\psi_p(x) &= 0, \\ V_p(x) &= 4E_p + x^6 + 2A_p\xi_p x^4 + [\xi_p(4M + 4\kappa_p + 2) + A_p^2]x^2 + \frac{(2\kappa_p - 1/2)(2\kappa_p - 3/2)}{x^2}, \end{aligned} \quad (12)$$

for the functions

$$\psi_p(x) = x^{2\kappa_p - \frac{1}{2}} \exp\left[\frac{x^2}{2} \left(A_p + \frac{\xi_p x^2}{2}\right)\right] Q_p(x),$$

$$Q_p(x) = \prod_{j=1}^{M_p} (x^2 - v_p^{(j)}).$$

Setting $\hat{E} = -4(E + \epsilon/2(M - q))$ and using the relations (8)–(11) we can write

$$-\psi_p''(x) + V_p(x)\psi_p(x) = \hat{E}\psi_p(x),$$

$$V_1(x) = x^6 + 6\epsilon x^4 + x^2[2(M - q + 1) + 9\epsilon^2] + \frac{(M + q + 1/2)(M + q + 3/2)}{x^2}, \quad (13)$$

$$V_2(x) = x^6 + 6\epsilon x^4 + x^2[-4M - 2(q + 2) + 9\epsilon^2] - 6\epsilon(M + 1) + \frac{(q + 1/2)(q - 1/2)}{x^2}, \quad (14)$$

and see that the $p = 1$ case shares the same QES spectrum as the $p = 2$ case.

Before proceeding further some remarks are required. Consider the general potential

$$V = x^6 + 2\delta x^4 + (\delta^2 + \alpha)x^2 + \frac{l(l + 1)}{x^2} + C, \quad (15)$$

where δ, α and l are real parameters and C is a real constant. Usually we require the wavefunction $\psi(x)$ to be square integrable along the positive real axis while behaving at the origin as $\psi|_{x \rightarrow 0} \sim x^{l+1}$. If $l > -1/2$ this defines a Hermitian problem and so we will use the subscript H to indicate this type of radial Schrödinger problem. We can alternatively consider PT -symmetric boundary conditions [6, 7], denoted by the subscript PT . In this case, we require the wavefunction to be square integrable along a contour in the complex plane \mathcal{C} which starts and ends at $|x| = \infty$ within the Stokes sectors defined as wedges in the complex plane of open angle $\pi/4$, centered at angles $-\pi/4$ and $\pi/4$. This contour should also avoid the origin whenever $l(l + 1) \neq 0$. In view of this we see that the $p = 1$ case of (12) corresponds to the PT -symmetric case while $p = 2$ is the Hermitian case.

We also mention that the change of variable $\epsilon \rightarrow -\epsilon$ is equivalent, up to a unitary transformation $a^\dagger \rightarrow -a^\dagger$, to the mapping $H \rightarrow -H$. Hence we also have the equivalence that the QES spectrum of $V_p(x; \epsilon)$ is the negative of the QES spectrum of $V_p(x; -\epsilon)$ for both $p = 1, 2$; this is the anti-isospectral duality of [31].

2.2. Equivalence beyond the QES spectrum

In this subsection, we will describe how the spectral equivalence obtained in the last section between the QES eigenvalues of a PT -symmetric and a Hermitian Schrödinger problem is in fact a complete spectral equivalence.

If we set $\alpha = \alpha_J = -(4J + 1 + 2l)$ and $C = -2\delta J$ in (15) then the results of the last section prove that the Schrödinger operators with potentials

$$V_H = x^6 + 2\delta x^4 + (\delta^2 - (4J + 1 + 2l))x^2 + \frac{l(l + 1)}{x^2} - 2\delta J, \quad \psi|_{x \rightarrow 0} \sim x^{l+1},$$

$$V_{PT} = x^6 + 2\delta x^4 + (\delta^2 + 2J - 1 - 2l)x^2 + \frac{(l + J)(l + J + 1)}{x^2}$$
(16)

are both quasi-exactly solvable and the $J = M + 1$ exactly known eigenvalues coincide.

The QES spectral equivalence (16) at $\delta = 0$ was discovered and proven [8] via the ODE/IM correspondence [1, 2]. In fact, the fifth spectral equivalence of [8] makes a stronger

Table 1. The spectrum of (17) and (18) with $\delta = 0.2, \alpha = 0.31, l = 0.54$.

n	E_n (Hermitian)	E_n (PT)
0	7.170 306 15	7.170 306 16
1	19.522 063 7	19.522 063 7
2	35.274 465 3	35.274 465 4
3	53.792 933 7	53.792 933 9
4	74.706 246 4	74.706 246 6

statement: it says that the full spectrum of the PT -symmetric and Hermitian problems (16) are isospectral, not just the QES levels. Moreover, the equivalence also holds away from the special QES points α_J . Numerically, we find this equivalence also extends to the case when $\delta \neq 0$. We conjecture that the most general equivalence is that

$$V_H = x^6 + 2\delta x^4 + (\delta^2 + \alpha)x^2 + \frac{l(l+1)}{x^2}, \quad \psi|_{x \rightarrow 0} \sim x^{l+1} \tag{17}$$

is isospectral to

$$V_{PT} = x^6 + 2\delta x^4 + \left(\delta^2 - \frac{1}{2}(\alpha + 6l + 3)\right)x^2 + \frac{(2l + 3 - \alpha)(2l - 1 - \alpha)}{16x^2} + \frac{\delta}{2}(\alpha + 1 + 2l). \tag{18}$$

The approach used in the previous section does not allow us to prove this statement away from the QES sector for the special points α_J . It would be interesting to generalize the approach of [8] in order to obtain a rigorous proof of this statement. Numerical confirmation of the spectral equivalence (17) and (18) for values of (δ, α, l) away from the QES points is shown in table 1.

2.3. Further spectral equivalences

Here we briefly comment that the further spectral equivalences obtained in [8] for (15) with $\delta = 0$ can also be generalized to the problem with $\delta \neq 0$.

The second equivalence of [8] relating the spectrum of two Hermitian sextic problems is easily generalized using the results of the last section. The PT -symmetric problem (18) is invariant under $(\delta, \alpha, l) \rightarrow (\delta, (6l + 3 - \alpha)/2, (\alpha + 2l - 1)/4)$, whereas the same transformation on the Hermitian case (17) leads to a different Hermitian problem. Rewriting, we obtain a full spectral equivalence between two radial problems:

$$\begin{aligned} V_H^1 &= x^6 + 2\delta x^4 + (\delta^2 + \alpha)x^2 + \frac{l(l+1)}{x^2} - \frac{\delta}{4}(1 + 2l - \alpha), \quad \psi|_{x \rightarrow 0} \sim x^{l+1}, \\ V_H^2 &= x^6 + 2\delta x^4 + \left(\delta^2 + \frac{1}{2}(3 - \alpha + 6l)\right)x^2 + \frac{(2l - 1 + \alpha)(2l + 3 + \alpha)}{16x^2} + \frac{\delta}{4}(1 + 2l - \alpha), \\ &\psi|_{x \rightarrow 0} \sim x^{(\alpha+2l-1)/4+1}. \end{aligned} \tag{19}$$

This equivalence can be proven in terms of Bender–Dunne polynomials [25]. Set

$$\psi(x, E, \delta, \alpha, l) = e^{-x^4/4 - \delta x^2/2} x^{l+1} \sum_{n=0}^{\infty} \left(\frac{-1}{4}\right)^n \frac{P_n(E, \delta, \alpha, l)}{n! \Gamma(n + l + 3/2)} x^{2n}. \tag{20}$$

To satisfy a radial ODE with general potential (15) the polynomials P_n must satisfy the three-term recursion relation

$$\begin{aligned} P_n(E) &= (E + C - \delta(2l + 4n - 1))P_{n-1}(E) \\ &+ 16(n - 1)(n + (\alpha + 2l - 3)/4)(n + l - 1/2)P_{n-2} \end{aligned}$$

Table 2. The spectrum of (24) and (25) with $\delta = 0.2, J = 3, l = 0.54$.

n	E_n (Hermitian)	E_n (PT)
0	-11.079 808 8	30.103 329 7
1	1.459 113 59	48.408 557 7
2	14.468 696 2	69.085 653 8
3	30.103 329 3	91.898 871 1
4	48.408 557 6	116.668 373

with $P_0(E) = 1, P_1(E) = E$. The wavefunction ψ is an everywhere-convergent series for all $l \neq -n - 3/2, n \in \mathbb{Z}$ for arbitrary α , with the required behavior at the origin built-in: that is, $\psi(x) \sim x^{l+1}$ at $x = 0$. For generic values of α , the wavefunction ψ will be an infinite series. However, whenever $\alpha = -(4J + 2l + 1)$ for positive integer J , the series truncates and the zeros of $P_J(E)$ are the QES eigenvalues [25]. The relevant point here is that the recursion relation is invariant under

$$\alpha \rightarrow (6l + 3 - \alpha)/2, \quad l \rightarrow (\alpha + 2l - 1)/4, \quad C \rightarrow C - \delta(1 + 2l - \alpha)/2, \quad (21)$$

giving rise, with $C = -\delta(1 + 2l - \alpha)/4$, to the spectral equivalence (19).

One more spectral equivalence may be obtained, either via the Bender–Dunne polynomials as in [8] or using Darboux–Crum transformations [26, 27], generalizing the third spectral equivalence of [8]. Setting $\alpha_J = -(4J + 2l + 1)$ we take the Hermitian QES problem and using a Darboux–Crum transformation remove all of the QES levels leaving the rest of the spectrum in place. The resulting potential is once again exactly the same as the original form modulo a change in the parameters. That is, the potential

$$V_H(x) = x^6 + 2\delta x^4 + (\delta^2 - (4J + 2l + 1))x^2 + \frac{l(l + 1)}{x^2}, \quad \psi|_{x \rightarrow 0} \sim x^{l+1} \quad (22)$$

is isospectral to

$$V_H(x) = x^6 + 2\delta x^4 + (\delta^2 + (2J - 2l - 1))x^2 + \frac{(l + J)(l + J + 1)}{x^2} + 2\delta J, \quad \psi|_{x \rightarrow 0} \sim x^{l+J+1} \quad (23)$$

except for the first J QES levels of (22). The result is unexpected because the intermediate potentials, found by removing one energy level at a time, are in general singular potentials. Note that this result can also be obtained from the cubic case of the type A \mathcal{N} -fold supersymmetry of [32].

Finally, combining the above equivalences, we find that the QES problem

$$V_H = x^6 + 2\delta x^4 + (\delta^2 - (4J + 2l + 1))x^2 + \frac{l(l + 1)}{x^2}, \quad \psi|_{x \rightarrow 0} \sim x^{l+1} \quad (24)$$

is isospectral to

$$V_{PT} = x^6 + 2\delta x^4 + (\delta^2 - (4J + 2l + 1))x^2 + \frac{l(l + 1)}{x^2} \quad (25)$$

except for the QES eigenvalues. Numerical confirmation is shown in table 2.

3. Spectral equivalences in hyperbolic potentials

Now we move on to examine another case in which a spectral equivalence can be established for potentials expressed in terms of hyperbolic functions. Here we consider the following

Hamiltonian:

$$H = \epsilon(n_1 + n_2 - n_3 - n_4) + g(n_1n_3 + n_2n_4 + a_1^\dagger a_2^\dagger a_3 a_4 + a_1 a_2 a_3^\dagger a_4^\dagger), \quad (26)$$

where the $\{a_j, a_j^\dagger \mid j = 1, 2, 3, 4\}$ are canonical boson operators and $n_j = a_j^\dagger a_j$. We note that the unitary transformation

$$a_1^\dagger \longleftrightarrow a_3^\dagger, \quad a_2^\dagger \longleftrightarrow a_4^\dagger \quad (27)$$

is equivalent to the change of variable

$$\epsilon \rightarrow -\epsilon. \quad (28)$$

We make the following assignment of the $su(2)$ operators with a central extension:

$$\begin{aligned} S_1^+ &= a_1^\dagger a_2^\dagger, & S_1^- &= -a_1 a_2, & S_1^z &= \frac{1}{2}(n_1 + n_2 + 1), & K_1 &= \frac{1}{2}(n_1 - n_2), \\ S_2^+ &= -a_3^\dagger a_4^\dagger, & S_2^- &= a_3 a_4, & S_2^z &= \frac{1}{2}(n_3 + n_4 + 1), & K_2 &= \frac{1}{2}(n_3 - n_4), \\ S_3^+ &= a_1^\dagger a_4, & S_3^- &= a_4^\dagger a_1, & S_3^z &= \frac{1}{2}(n_1 - n_4), & K_3 &= \frac{1}{2}(n_1 + n_4), \\ S_4^+ &= a_3^\dagger a_2, & S_4^- &= a_2^\dagger a_3, & S_4^z &= \frac{1}{2}(n_3 - n_2), & K_4 &= \frac{1}{2}(n_2 + n_3), \end{aligned}$$

which for $j = 1, 2, 3, 4$ satisfy the commutation relations:

$$[S_j^z, S_j^\pm] = \pm S_j^\pm, \quad [S_j^+, S_j^-] = 2S_j^z, \quad [K_j, S_j^\pm] = [K_j, S_j^z] = 0.$$

We will also need the corresponding Casimir invariants

$$C_j = S_j^+ S_j^- + S_j^z (S_j^z - I).$$

In terms of these operators we can express the Hamiltonian (26) as

$$H = H_\alpha - g(C_1 + C_2 - (S_1^z + S_2^z)^2 - 2K_1 K_2 + 2(S_1^z + S_2^z) - \frac{1}{2}I) \quad (29)$$

$$= H_\beta - g(C_3 + C_4 - (S_3^z + S_4^z)^2 - 2K_3 K_4 + S_3^z + S_4^z). \quad (30)$$

where

$$H_\alpha = 2\epsilon(S_1^z - S_2^z) + g(S_1^+ S_1^- + S_2^+ S_2^- + S_1^+ S_2^- + S_2^+ S_1^-),$$

$$H_\beta = 2\epsilon(S_3^z - S_4^z) + g(S_3^+ S_3^- + S_4^+ S_4^- + S_3^+ S_4^- + S_4^+ S_3^-).$$

We recognize from [30] (see equation (69)) that H_α, H_β are Bethe ansatz solvable. Since the additional terms appearing in (29) commute with H_α , we can extend it to a solution for H . Likewise, since the terms in (30) commute with H_β , we can also obtain a second solution from this expression.

To obtain the Bethe ansatz solutions we need to identify a reference state $|\phi\rangle$ which satisfies

$$S_j^- |\phi\rangle = 0, \quad S_j^z |\phi\rangle = -s_j |\phi\rangle$$

for some scalars s_j ; i.e. $|\phi\rangle$ is a lowest weight state for all realizations of the $su(2)$ algebra. The choices

$$|\phi(p_\sigma, q_\sigma)\rangle = \frac{1}{\sqrt{p_\sigma! q_\sigma!}} (a_2^\dagger)^{p_\sigma} (a_4^\dagger)^{q_\sigma} |0\rangle, \quad (31)$$

$\sigma = \alpha, \beta$ satisfy this condition with

$$s_1 = -\frac{p_\alpha + 1}{2}, \quad s_2 = -\frac{q_\alpha + 1}{2}, \quad s_3 = \frac{q_\beta}{2}, \quad s_4 = \frac{p_\beta}{2}.$$

Now we can write the exact solution from [30]. For the representation (29) we have that the energies of H_α are given by

$$E_\alpha = (p_\alpha - q_\alpha)\epsilon - 2 \sum_{j=1}^{M_\alpha} v_\alpha^{(j)} \quad (32)$$

where the $\{v_\alpha^{(j)}\}$ are solutions of the BAE

$$\frac{2}{g} - \frac{p_\alpha + 1}{v_\alpha^{(j)} + \epsilon} - \frac{q_\alpha + 1}{v_\alpha^{(j)} - \epsilon} = \sum_{k \neq j}^M \frac{2}{v_\alpha^{(j)} - v_\alpha^{(k)}}, \quad j = 1, \dots, M_\alpha. \quad (33)$$

For a given Bethe root, the eigenstate is given by

$$|\{v_\alpha^{(j)}\}\rangle = \prod_{k=1}^{M_\alpha} \left(\frac{S_1^+}{v_\alpha^{(k)} + \epsilon} + \frac{S_2^+}{v_\alpha^{(k)} - \epsilon} \right) |\phi(p_\alpha, q_\alpha)\rangle. \quad (34)$$

Using this explicit form for the eigenstates we then deduce that

$$\begin{aligned} C_1 |\{v_\alpha^{(j)}\}\rangle &= \frac{(p_\alpha^2 - 1)}{4} |\{v_\alpha^{(j)}\}\rangle \\ C_2 |\{v_\alpha^{(j)}\}\rangle &= \frac{(q_\alpha^2 - 1)}{4} |\{v_\alpha^{(j)}\}\rangle \\ (S_1^z + S_2^z) |\{v_\alpha^{(j)}\}\rangle &= \left(M_\alpha + 1 + \frac{p_\alpha + q_\alpha}{2} \right) |\{v_\alpha^{(j)}\}\rangle \\ K_1 K_2 |\{v_\alpha^{(j)}\}\rangle &= \frac{p_\alpha q_\alpha}{4} |\{v_\alpha^{(j)}\}\rangle. \end{aligned}$$

Hence the energy of the Hamiltonian (29) in terms of the Bethe roots $\{v_\alpha^{(j)}\}$ is

$$E = g(p_\alpha + M_\alpha)(q_\alpha + M_\alpha) + (p_\alpha - q_\alpha)\epsilon - 2 \sum_{j=1}^{M_\alpha} v_\alpha^{(j)}. \quad (35)$$

For the representation (30) the energies of the Hamiltonian H_β are as follows,

$$E_\beta = (p_\beta - q_\beta)\epsilon - 2 \sum_{j=1}^{M_\beta} v_\beta^{(j)}, \quad (36)$$

where the $\{v_\beta^{(j)}\}$ are solutions of the BAE

$$\frac{2}{g} + \frac{q_\beta}{v_\beta^{(j)} + \epsilon} + \frac{p_\beta}{v_\beta^{(j)} - \epsilon} = \sum_{k \neq j}^{M_\beta} \frac{2}{v_\beta^{(j)} - v_\beta^{(k)}}, \quad j = 1, \dots, M_\beta. \quad (37)$$

For a given solution, the eigenstate is given by

$$|\{v_\beta^{(j)}\}\rangle = \prod_{k=1}^{M_\beta} \left(\frac{S_3^+}{v_\beta^{(k)} + \epsilon} + \frac{S_4^+}{v_\beta^{(k)} - \epsilon} \right) |\phi(p_\beta, q_\beta)\rangle. \quad (38)$$

Note that if

$$M_\beta > \min(p_\beta, q_\beta),$$

then (38) vanishes. This is in contrast to (34), for which there is no analogous constraint on M_α . Taking the above form for the eigenstates, we find that

$$\begin{aligned} C_3|\{v_\beta^{(j)}\}\rangle &= \frac{q_\beta(q_\beta + 2)}{4}|\{v_\beta^{(j)}\}\rangle \\ C_4|\{v_\beta^{(j)}\}\rangle &= \frac{p_\beta(p_\beta + 2)}{4}|\{v_\beta^{(j)}\}\rangle \\ (S_3^z + S_4^z)|\{v_\beta^{(j)}\}\rangle &= \left(M_\beta - \frac{p_\beta + q_\beta}{2}\right)|\{v_\beta^{(j)}\}\rangle \\ K_3K_4|\{v_\beta^{(j)}\}\rangle &= \frac{p_\beta q_\beta}{4}|\{v_\beta^{(j)}\}\rangle. \end{aligned}$$

So that the corresponding energy for the Hamiltonian (30) in terms of the Bethe roots $\{v_\beta^{(j)}\}$ is

$$E = g(M_\beta - p_\beta)(M_\beta - q_\beta) - gM_\beta + (p_\beta - q_\beta)\epsilon - 2 \sum_{j=1}^{M_\beta} v_\beta^{(j)}. \quad (39)$$

In order to compare the two Bethe ansatz solutions we need to determine the relationship between the parameters $\{p_\alpha, q_\alpha, M_\alpha\}$ and $\{p_\beta, q_\beta, M_\beta\}$. From the form of the eigenstates (34), (38) it is deduced that

$$\begin{aligned} n_1 + n_4 &= q_\alpha + M_\alpha = q_\beta, \\ n_2 + n_3 &= p_\alpha + M_\alpha = p_\beta, \\ n_1 - n_2 &= -p_\alpha = M_\beta - p_\beta, \end{aligned}$$

giving the solution

$$q_\beta = q_\alpha + M_\alpha, \quad p_\beta = p_\alpha + M_\alpha, \quad M_\beta = M_\alpha.$$

Hence for every solution of (33) with $\{p_\alpha, q_\alpha, M_\alpha\}$ giving energy E via (35) there is a solution of (37) with $\{p_\beta = p_\alpha + M_\alpha, q_\beta = q_\alpha + M_\alpha, M_\beta = M_\alpha\}$ giving the same energy E via (39). Note that in this correspondence the constraint (39) is never violated. Our next goal is to use this result to establish a spectral equivalence in the QES sector of a Schrödinger equation.

3.1. Equivalences of QES sectors

We start with the general form of Bethe ansatz equations

$$A + \frac{B}{v^{(j)} + \gamma/2} + \frac{C}{v^{(j)} - \gamma/2} = \sum_{k \neq j}^M \frac{2}{v^{(j)} - v^{(k)}} \quad (40)$$

for $A, B, C \in \mathbb{R}$ and set

$$\begin{aligned} \psi(x) &= (\cosh(x) - 1)^{-(B/2+1/4)} (\cosh(x) + 1)^{-(C/2+1/4)} \\ &\times \exp\left(\frac{A\gamma}{4} \cosh(x)\right) \prod_{j=1}^M \left(\frac{\gamma}{2} \cosh(x) + v^{(j)}\right). \end{aligned} \quad (41)$$

It can be shown [34] that $\psi(x)$ satisfies the Schrödinger equation

$$-\frac{d^2\psi}{dx^2} + V(x)\psi = \mathcal{E}\psi, \quad (42)$$

where

$$V(x; A, B, C, \gamma) = M \left(M - B - C + \frac{A\gamma}{2} \cosh(x) - 1 \right) + \frac{1}{4}(B + C + 1)^2 + \frac{A^2\gamma^2}{16} \sinh^2(x) + \frac{A\gamma(C - B)}{4} - \frac{A\gamma(B + C)}{4} \cosh(x) + \frac{(2B + 1)(2B + 3)}{8(\cosh(x) - 1)} - \frac{(2C + 1)(2C + 3)}{8(\cosh(x) + 1)},$$

$$\mathcal{E} = -A \sum_{j=1}^M v^{(j)}.$$

We note that this potential has the symmetry

$$V(x; A, B, C, \gamma) = V(x + i\pi; A, C, B, -\gamma). \tag{43}$$

Assuming $A\gamma$ is negative, we see that there is a spectral equivalence between the Hermitian problem with potential $V(x; A, B, C, \gamma)$ and the PT -symmetric problem $V(x; A, C, B, -\gamma)$ defined on the contour $\Im(x) = \pi$. Specifically, there is no PT -symmetry breaking in the latter.

Returning to the Hamiltonian (26), we fix $g = 1$. Because the Bethe ansatz equations (33), (37) are of the same form as (40), we can map the spectrum of (26) to that of the QES sector of (42) by adding the appropriate terms to the potential. Since there are two Bethe ansatz solutions for (26) we obtain the potentials

$$V_\alpha(x; p_\alpha, q_\alpha) = V(x; 2, -(p_\alpha + 1), -(q_\alpha + 1), 2\epsilon) + (p_\alpha + M_\alpha)(q_\alpha + M_\alpha) + (p_\alpha - q_\alpha)\epsilon,$$

$$V_\beta(x; p_\beta, q_\beta) = V(x; 2, q_\beta, p_\beta, 2\epsilon) + (M_\beta - p_\beta)(M_\beta - q_\beta) - M_\beta + (p_\beta - q_\beta)\epsilon,$$

where for $V_\alpha, \mathcal{E} = -2 \sum_{j=1}^{M_\alpha} v_\alpha^{(j)}$ while for $V_\beta, \mathcal{E} = -2 \sum_{j=1}^{M_\beta} v_\beta^{(j)}$. We have already seen that the Bethe ansatz solutions for (26) are equivalent when $M_\alpha = M_\beta, p_\beta = p_\alpha + M_\alpha, q_\beta = q_\alpha + M_\alpha$. It follows that the potentials $V_\alpha(x; p_\alpha, q_\alpha)$ and $V_\beta(x; (p_\alpha + M_\alpha), (q_\alpha + M_\alpha))$ have the same QES spectrum.

However, the QES wavefunctions of the Schrödinger equation (42) are not always normalizable on the full real line. The potential $V(x; A, B, C, \gamma)$ has a singularity at the origin whenever $(2B + 1)(2B + 3) \neq 0$. When $B = -3/2$ or $-1/2$, the potential is nonsingular and the QES wavefunctions (41) (assuming $A\gamma$ is negative) can be extended to normalizable odd/even wavefunctions respectively on the full real line [33].

It is interesting to consider the two nonsingular cases. The above result with $(p_\alpha, q_\alpha) = (-1/2, -3/2 - M)$ establishes a QES spectral equivalence between a potential with $B = -3/2$ and a potential with $B = -1/2$. By adding constant shifts to $V_\alpha(x; p_\alpha, q_\alpha)$ and $V_\beta(x; (p_\alpha + M_\alpha), (q_\alpha + M_\alpha))$, we can prove that the spectral equivalence extends to the full spectrum, except for the presence of a single $E = 0$ energy level in the former. In fact, the potentials are supersymmetric partners [28, 29]. Set

$$\mathcal{Q}_\pm(x) = \pm \frac{d}{dx} + \frac{(M + 1) \sinh x}{2(\cosh x + 1)} + \frac{\gamma}{2} \sinh(x),$$

then

$$\mathcal{Q}_+ \mathcal{Q}_- \psi(x) = \left[-\frac{d^2}{dx^2} + V(x; 2, -3/2, M - 1/2, \gamma) - \gamma(M + 1) \right] \psi(x) = E \psi(x),$$

with corresponding $E = 0$ eigenfunction

$$\psi(x) = (\cosh x + 1)^{(M+1)/2} \times \exp\left(\frac{\gamma}{2} \cosh x\right).$$

We immediately deduce that the supersymmetric partner

$$Q_- Q_+ \Psi(x) = E \Psi(x)$$

has potential $V(x; 2, -1/2, M + 1/2, \gamma) + M - \gamma(M + 1)$. We have therefore established complete isospectrality between these Schrödinger problems, up to the $E = 0$ energy level. Moreover, this spectral equivalence holds for all real values of M .

Finally, we remark that the unitary transformation (27) combined with the change of variable (28) leaves the Hamiltonian invariant. Observe that (27) applied to (31) has the effect that

$$p \longleftrightarrow q. \quad (44)$$

This is reflected in the symmetry of the Bethe ansatz equations (33), (37) which are invariant under the combination of (28) and (44). Thus the unitary transformation (27) effectively interchanges the Hermitian and PT -symmetric versions of the potentials $V_\alpha(x; p_\alpha, q_\alpha)$, $V_\beta(x; p_\beta, q_\beta)$, which explains the existence of the symmetry (43).

This scenario is somewhat different to the case of the QES sextic potential discussed previously. There, the unitary transformation maps the QES spectrum into the negative of the QES spectrum. The equivalence between the Hermitian and PT -symmetric problems is due to the equivalence of the Bethe ansatz solutions. For the above case, the unitary transformation maps between the Hermitian and PT -symmetric cases, while the equivalence of the Bethe ansatz solutions gives a spectral equivalence between two Hermitian QES potentials.

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